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CONTENTS:

PROJECTIVE TRANSFORMATIONS IN ONE DIMENSION AND THEIR CONTINUOUS GROUPS, ... *H. B. Newson.*

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PROJECTIVE TRANSFORMATIONS IN ONE DIMENSION AND THEIR CONTINUOUS GROUPS.

BY H. B. NEWSON.

INTRODUCTION.

THE writer of this paper is engaged in establishing and developing a new theory of collineations and their Lie groups. A recent memoir in the *American Journal of Mathematics*, Vol. XXIV, p. 109, entitled "A New Theory of Collineations and their Lie Groups," treats of collineations in a plane. A series of papers applying the theory to collineations in space has been published in the Kansas University Quarterly, Vol. X. This theory of collineations in two, three and higher dimensions is based upon, and presupposes, a corresponding theory of one-dimensional projective transformations.

In the present paper the theory is developed for the range of points on a line, but it applies equally well to all three one-dimensional primary forms of projective geometry, viz., the range of points on a line, the pencil of lines through a point, and the pencil of planes through a line. The object of the paper is to collect and set forth the principal facts of one-dimensional transformations, and to build thereon a comprehensive theory of their continuous groups. The paper is sufficiently complete to serve as a foundation on which to build a consistent theory of collineations in two, three and higher dimensions. In form and content it is suitable for the beginner, and will serve as an introduction to the general theory.

The chief sources are as follows: In his "Vorlesungen über Continuierliche Gruppen," Kapitel 5, Lie gives a detailed account of his theory of one-dimensional projective transformations and their continuous groups. His equation is,

$$x_1 = \frac{ax+b}{cx+d}, \qquad (1)$$

where all the variables and coefficients are assumed to be complex

numbers. His geometrical interpretation is by points on a line, both points and line being, in general, imaginary.

Klein,* Poincaré† and others have extensively investigated the same transformation, but chiefly with respect to its discontinuous groups. They have interpreted their results geometrically by points in the complex plane. In addition to the results of Lie and Klein, the principal results of two former papers by the writer are incorporated in the present memoir. These are entitled "Continuous Groups of Projective Transformations Treated Synthetically,"‡ and "Continuous Groups of Circular Transformations."§ In the first of these papers is developed geometrically the elements of this new theory of projective transformations in one dimension. In the second paper the continuous groups of the transformations given by (1) are determined and interpreted as transformations of the points of a complex plane.

PART I.

§1. General Properties of One-dimensional Transformations.

The resultant of two transformations.—A projective transformation should be looked upon as an operation which, when applied to a finite set of points on a line or to the totality of all points on the line, has the effect of rearranging and redistributing them in such a way that the new set or range is projective with the old. If we carry out in succession two projective transformations on a set of points, the result is equivalent to a single projective transformation. This may be shown as follows:

Let T and T_1 be two transformations whose equations are respectively

$$x_1 = \frac{ax+b}{cx+d} \text{ and } x_2 = \frac{a_1x_1+b_1}{c_1x_1+d_1}.$$

The first transforms the point x into x_1, and the second transforms x_1 into x_2. We suppose the operations are carried out in the order in which the equations are written. If we eliminate x_1 from the above, we get

$$x_2 = \frac{(a_1a+b_1c)\,x+(a_1b+b_1d)}{(c_1a+d_1c)\,x+(c_1b+d_1d)}. \qquad (2)$$

It should be observed that (2) is of the same form as (1) and differs from it only in the value of the coefficients. Equation (2) therefore expresses a projective transformation T_2, which transforms the point x directly to x_2, and is equivalent to the successive applications of T and T_1 in the order named. The transformation T_2 is

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*See Modulfunctionen, Band I, S. 163–207.

†See Acta Mathematica, Tome I, pp. 1–62.

‡Kansas University Quarterly, Vol. IV, pp. 71–93.

§Bulletin Amer. Math. Soc., 2d Series, Vol. IV, pp. 107–121.

called the resultant of the transformations T and T_1, which are called the component transformations. The operation is symbolized thus: $TT_1 = T_2$. If the two component transformations T and T_1 are taken in the inverse order, the resultant, $T_1T = T'_2$, is not the same as T_2. Thus:

$$T'_2: \quad x_2 = \frac{(aa_1 + bc_1)\,x + (ab_1 + bd_1)}{(ca_1 + dc_1)\,x + (cb_1 + dd_1)}, \qquad (3)$$

which is not the same as T_2. The two projective transformations T_2 and T'_2 are called *conjugate* transformations.

Determinant of a transformation.—The determinant

$$\triangle \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is called the determinant or modulus of the transformation (1), and it is assumed, for the present at least, that $\triangle$ does not vanish.

By referring to the transformations lettered T, T_1, T_2, we see that the determinant of T_2 is

$$\begin{vmatrix} a_1a + b_1c & a_1b + b_1d \\ c_1a + d_1c & c_1b + d_1d \end{vmatrix};$$

but this is the product of

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ by } \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix};$$

these determinants are respectively the determinants of T and T_1, the components of T_2. Hence the determinant of a transformation, T_2, which is the resultant of the transformations T and T_1, is equal to the product of the determinants of T and T_1.

This result is capable of immediate extension; for let T_a, T_b and T_c denote three transformations, the result of whose successive applications is equivalent to T_d; the compounding of T_a and T_b is equivalent to a third transformation, T_{ab}. The resultant of T_{ab} and T_c is T_d, and the determinant of T_d is equal to the product of the determinants of T_{ab} and T_c; hence the determinant of T_d is equal to the product of those of T_a, T_b, and T_c. This mode of reasoning is applicable to the resultant of any number of transformations; hence by induction we infer the following theorem:

THEOREM 1. The resultant T_n of n projective transformations T_i ($i = 0, 1, 2, \ldots n-1$) is a projective transformation, and the determinant of the resultant is equal to the product of the determinants of the components.

Inverse and identical transformations.—The transformation T expressed by

$$x_1 = \frac{ax + b}{cx + d}$$

transforms the point x into x_1. This equation may be solved for x, giving us

$$x = \frac{-dx_1 + b}{cx_1 - a}. \qquad (4)$$

The transformation expressed by this equation is called the *inverse* of T, and is symbolized by T^{-1}. T^{-1} transforms a point x_1 into x. The resultant of T and T^{-1} is geometrically evident; it leaves every point on the line unchanged, and is therefore called an *identical* transformation. The resultant of T and T^{-1} reduces to $x_2 = x$, which shows that every point on the line is transformed into itself.

Invariant points.—When the points of a line are shifted into new positions by a projective transformation T, does it ever happen that one or more of the points are unaltered in position? To answer this question, we reason as follows: The coordinate of a point x, which remains at rest or unaltered in position, *i. e.*, which is transformed into itself, must satisfy the equation

$$x = \frac{ax + b}{cx + d}.$$

Clearing of fraction, we see that the coordinates of all such points satisfy the quadratic equation

$$cx^2 + (d - a)x - b = o;$$

whence we conclude that a projective transformation T leaves unaltered two points on the line, and their coordinates are given by the roots of equation (5). These two points are generally distinct, but for special values of a, b, c, d, they may coincide. They are called the invariant points of the transformation. Two transformations will not generally have the same invariant points, but, as we shall learn, an unlimited number of transformations may have one or both invariant points in common.

THEOREM 2. Every projective transformation of the points on a line, except the identical transformation, leaves invariant two points which are distinct or coincident.

Pseudo-transformations.—If the determinant of T vanishes, the transformation is called a pseudo-transformation. In defining the transformation, it was expressly stated that the determinant must not be zero. This condition excludes just these transformations called pseudo-transformations. The equation of the transformation is written

$$x_1 = \frac{ax + b}{cx + d};$$

if the determinant $ad - bc = o$, then $d = \frac{bc}{a}$. Substituting this value of d in the equation, we have

$$x_1 = \frac{a(ax + b)}{c(ax + b)} = \frac{a}{c}; \qquad (6)$$

which shows that every point on the line is transformed into the fixed point $\frac{a}{c}$. The inverse of the transformation T is written

$$x = \frac{-dx_1 + b}{cx_1 - a}.$$

The determinant of this is also $ad - bc$, which equated to zero also gives $d = \frac{bc}{a}$. Substituting this value of d in the last equation, we have

$$x = -\frac{b(cx-a)}{a(cx-a)} = -\frac{b}{a}; \tag{7}$$

which shows that every point on the line is transformed by (7) into the fixed point $-\frac{b}{a}$.

The invariant points of a pseudo-transformation are also given by equation (5). Putting $d = \frac{bc}{a}$ in this equation, it breaks up into

$$\left(x - \frac{a}{c}\right)\left(x + \frac{b}{a}\right) = o; \tag{8}$$

thus showing that $\frac{a}{c}$ and $-\frac{b}{a}$ are the invariant points of the pseudo-transformation.

THEOREM 3. A pseudo-transformation transforms every point on the line into one or the other of its invariant points.

Three conditions determine a projective transformation.—The equation of a projective transformation T contains three independent constants, viz., $a : b : c : d$. We infer, therefore, that three conditions determine such a transformation. In particular, three points and their corresponding points determine uniquely and completely a projective transformation.

Let x', x'', x''' be any three points on a line, and x_1', x_1'', x_1''' their corresponding points, respectively. Substituting successively in (1) the coordinates of each pair of corresponding points, we have three equations, viz.:

$$\begin{aligned} cx'x'_1 + dx'_1 - ax' - b &= o, \\ cx''x''_1 + dx''_1 - ax'' - b &= o, \\ cx'''x'''_1 + dx'''_1 - ax''' - b &= o. \end{aligned} \tag{9}$$

These equations are linear and homogeneous in a, b, c, d, and determine the ratios of these quantities uniquely and completely, provided no two of these equations are identical or have their coefficients proportional.

THEOREM 4. There is one and only one projective transformation that transforms three given points on a line into three other given points.

The identical transformation.—Suppose that the transformation (1) leaves three points of the line invariant. If we put $x'_1 = x'$, $x''_1 = x''$ and $x'''_1 = x'''$ in equations (9), these reduce to the following:

$$\begin{aligned} cx'^2 + (d-a)x' - b &= o, \\ cx''^2 + (d-a)x'' - b &= o, \\ cx'''^2 + (d-a)x''' - b &= o. \end{aligned} \tag{10}$$

The determinant of these equations,

$$\begin{vmatrix} x'^2 & x' & 1 \\ x''^2 & x'' & 1 \\ x'''^2 & x''' & 1 \end{vmatrix} \equiv (x'-x''')(x''-x''')(x'-x''), \tag{11}$$

does not vanish so long as the three points are distinct; consequently, the coefficients of the above equations must vanish identically. Thus, $c = o$, $b = o$, $d = a$. Putting these values in (1) we get $x_1 = x$, which is the identical transformation. The identical transformation we know transforms every point of the line into itself.

THEOREM 5. A projective transformation which leaves three points of a line invariant is the identical transformation and leaves all points of the line invariant.

§ 2. TYPES AND NORMAL FORMS OF PROJECTIVE TRANSFORMATIONS.

Two types of projective transformations.—The invariant points of a transformation T are given by the roots of the quadratic equation (5). The roots of this equation are:

$$(A, A') = \frac{a - d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2c}. \tag{12}$$

These two roots are distinct or coincident, according as

$$(a+d)^2 - (ad - bc)$$

is not or is equal to zero. Thus there are two distinct types of transformation. The first type is characterized by the fact that it has two invariant points, while the second type has only one. Every transformation not identical belongs to one or the other of these types.

Normal form of type I.—A transformation T of type I, whose invariant points are A and A′, may be written in the form:

$$\frac{x_1 - A'}{x_1 - A} = k\,\frac{x - A'}{x - A}, \tag{13}$$

where the constant k is expressible in the terms of the coefficients a, b, c, d, as follows:

$$k = \frac{(a + d + \sqrt{(a+d)^2 - 4(ad - bc)})^2}{4(ad - bc)}. \tag{14}$$

In order to verify this, we must solve equation (13) for x_1; this gives us

$$x_1 = \frac{(A - kA')x - AA'(1-k)}{(1-k)x + (kA - A')}, \tag{15}$$

which is of the same form as

$$x_1 = \frac{ax+b}{cx+d}.$$

Comparing the coefficients of these forms, we have

$$\frac{A-kA'}{1-k} = \frac{a}{c},\ AA' = -\frac{b}{c},\ \frac{kA-A'}{1-k} = \frac{d}{c};$$

solving for A, A′ and k, we find

$$A = \frac{a-d+\sqrt{(a+d)^2-4(ad-bc)}}{2c},$$

$$A' = \frac{a-d-\sqrt{(a+d)^2-4(ad-bc)}}{2c}, \qquad (16)$$

$$k = \frac{(a+d+\sqrt{(a+d)^2-4(ad-bc)})^2}{4(ad-bc)}.$$

The values of A and A′ thus obtained are the same as the roots of equation (5). Equation (13) is called the *implicit* normal form of type I.

Normal form of type II.—A transformation of type II, whose single invariant point is A, is reducible to the form

$$\frac{1}{x_1-A} = \frac{1}{x-A} + t. \qquad (17)$$

To verify this, solve for x_1; thus,

$$x_1 = \frac{(1+tA)x - t^2A}{tx+(1-tA)}. \qquad (18)$$

This is the same form as (1). A and t are found in terms of a, b, c, d, as before, by comparing coefficients and solving for A and t; thus,

$$A = \frac{a-d}{2c} \text{ and } t = \frac{2c}{a+d}. \qquad (19)$$

Equation (17) is called the implicit normal form of type II.

THEOREM 6. Every transformation of the form $x_1 = \frac{ax+b}{cx+d}$ is reducible to one or the other of the normal forms

$$\frac{x_1-A'}{x_1-A} = k\frac{x-A'}{x-A} \text{ or } \frac{1}{x_1-A} = \frac{1}{x-A} + t.$$

Geometrical interpretation of the normal forms.—The normal form of type I may be written:

$$k = \frac{x-A}{x-A'} : \frac{x_1-A}{x_1-A'} = (AA'xx_1); \qquad (20)$$

i. e., k is the cross-ratio of the four points A, A′, x, x_1, where A and A′ are the invariant points, and x and x_1 a pair of corresponding points. Here x and x_1 are any pair of corresponding points, and k is a constant quantity.

In the normal form of type II the expressions x—A and x_1—A are the distances of a pair of corresponding points from the invariant point. The normal form of type II may be written:

$$\frac{1}{x_1-A} - \frac{1}{x-A} = t, \qquad (21)$$

which shows that the difference of the reciprocals of the distances of

a pair of corresponding points from the invariant point is constant for all pairs of corresponding points. Let x be the point at infinity on the line; t is thus seen to be the reciprocal of the segment Ax_1 where x_1 is the point into which the point at infinity is transformed.

THEOREM 7. In a transformation of type I, k, the cross-ratio of the invariant points and a pair of corresponding points, is constant for all pairs of corresponding points; in a transformation of type II, t, the difference of the reciprocals of the distances of a pair of corresponding points, is constant for all pairs of corresponding points.

The natural parameters.—When the transformation is written in the form of equation (1), we see that there are three independent parameters viz., $\frac{a}{d}$, $\frac{b}{d}$, $\frac{c}{d}$, when it is of type I; in the case of a transformation of type II, the relation, $(a+d)^2=4(ad-bc)$, is satisfied, and there are but two independent parameters. The coefficients, a, b, c, d, have no simple geometric meanings; but in the normal forms A, A′, k and A, t have definite important geometric meanings. The parameters A, A′, k and A, t are called the *natural* parameters of the transformation.

Explicit normal forms.—Equations (15) and (18) may be put into the forms:

$$x_1=\frac{\begin{vmatrix} x & 1 & 0 \\ A & 1 & A \\ A' & 1 & kA' \end{vmatrix}}{\begin{vmatrix} x & 1 & 0 \\ A & 1 & 1 \\ A' & 1 & k \end{vmatrix}}, \text{ and } x_1=\frac{\begin{vmatrix} x & 1 & 0 \\ A & 1 & A \\ 1 & 0 & tA+1 \end{vmatrix}}{\begin{vmatrix} x & 1 & 0 \\ A & 1 & 1 \\ 1 & 0 & t \end{vmatrix}}. \qquad (22)$$

These are called the explicit normal forms of types I and II, respectively.

Type II as the limiting form of type I.—It is evident that type II is the limiting form of type I when the two invariant points coincide. From equation (16) we see that $k=1$ when $A=A'$. The fraction $\frac{k-1}{A'-A}$ becomes indeterminate when $A=A'$. Putting for A, A′ and k their values from (16), we have:

$$\lim_{A'=A}\frac{k-1}{A'-A}=\frac{2c}{a+d}.$$

But from (19) $\frac{2c}{a+d}=t$; hence, $t=\lim_{A'=A}\frac{k-1}{A'-A}$. (23)

By means of this relation the normal form of type II can be deduced directly from that of type I. Dividing both numerator and denominator of (15) by $A-A'$, we get:

$$x_1=\frac{\frac{(A-kA')}{(A-A')}x-\frac{AA'(1-k)}{(A-A')}}{\frac{(1-k)}{(A-A')}x+\frac{kA-A'}{(A-A')}}.$$

Putting $A' = A$ and $\lim_{A'=A} \frac{1-k}{A-A'} = t$, this reduces to (18).

In the explicit normal form of type I, (22), subtract the second row from the last, divide through by $A'-A$, and pass to the limit. In this way we get the explicit normal form of type II.

Resultant of T and T_1.—We next consider the resultant of two transformations T and T_1, both of type I, which have no invariant point in common, and which are given in their explicit normal forms. T and T_1 are as follows:

$$x_1 = \frac{(A-kA')x - AA'(1-k)}{(1-k)x-(A'-kA)} \text{ and } x_2 = \frac{(A_1-k_1A'_1)x_1 - A_1A'_1(1-k_1)}{(1-k_1)x_1-(A'_1-k_1A_1)}.$$

Eliminating x_1 we have:

$$x_2 = \frac{\left\{(A-kA')\ (A_1-k_1A'_1) - AA'(1-k)\ (1-k_1)\right\}x}{\left\{\ (1-k)\ (A-kA') - (1-k_1)\ (A'_1-k_1A_1)\ \right\}x}$$
$$- \frac{\left\{(AA'(1-k)\ (A_1-k_1A'_1) - A_1A'_1(1-k_1)\ (A'-kA')\right\}}{-\left\{\ AA'(1-k)\ (1-k_1) - (A'-kA)\ (A'_1-k_1A_1)\ \right\}}$$
$$= \frac{px-q}{rx-s} = \frac{(A_2-k_2A'_2)x - A_2A'_2(1-k_2)}{(1-k_2)x-(A'_2-k_2A_2)}. \tag{24}$$

We readily find:

$$A_2A'_2 = \frac{q}{r},\ A_2+A'_2 = \frac{p+s}{r},\ A_2-A'_2 = \frac{1-k_2}{1+k_2}\cdot\frac{p-s}{r},$$
$$A_2-A'_2 = \frac{\sqrt{(p+s)^2-4\,qr}}{r}.$$

Hence,

$$\frac{1-k_2}{1+k_2}(p-s) = \sqrt{(p+s)^2-4\,qr}.$$

From this we find:

$$\frac{1+k_2}{\sqrt{k_2}} = \frac{p-s}{\sqrt{qr-ps}} = \frac{(1-kk_1)\ (A-A'_1)\ (A'-A_1) + (k+k_1)\ (A-A_1)\ (A'-A'_1)}{\sqrt{kk_1}\ (A-A')\ (A_1-A'_1)}$$
$$= \frac{1-k-k_1+kk_1}{\sqrt{kk_1}}\,l + \frac{k+k_1}{\sqrt{kk_1}}, \tag{25}$$

where

$$l = \frac{A_1-A'}{A_1-A'_1} : \frac{A-A'}{A-A'_1} = (A'A'_1)A_1A,$$

one of the cross-ratios of the four points A, A'_1, A_1, A'.

We also find

$$A_2 = \frac{p+k_2s}{r(1+k_2)} \text{ and } A'_2 = \frac{s+k_2p}{r(1+k_2)}. \tag{26}$$

If one or both of the transformations T, T_1 are of type II, the resultant is obtained by putting $A'=A$ and $\lim\frac{1-k}{A-A'} = t$, or $A'_1 = A'$ and $\lim\frac{1-k_1}{A_1-A'_1} = t_1$, or both, as the case may be, in (24), (25), and (26.)

§ 3. One-parameter Groups of Projective Transformations.

Resultant of T and T_1 with common invariant points.—Let T and T_1 be two transformations of type I having the same invariant points A and A′, and let T transform the point x to x_1, and let T_1 transform x_1 to x_2. The resultant of T and T_1 also leaves A and A′ in-

variant and transforms x directly to x_2. Let T and T_1 be given in the implicit normal forms:

$$\frac{x_1 - A'}{x_1 - A} = k\frac{x - A'}{x - A} \text{ and } \frac{x_2 - A'}{x_2 - A} = k_1\frac{x_1 - A'}{x_1 - A}.$$

We eliminate x_1 from these equations by multiplication, and obtain:

$$\frac{x_2 - A'}{x_2 - A} = kk_1\frac{x - A'}{x - A}.$$

The cross-ratio of T_2 is therefore $k_2 = kk_1$.

In the same way it may be shown that the resultant of any number of transformations with the same invariant points has its cross-ratio equal to the continued product of the cross-ratios of the components.

The cross-ratio k may have an infinite number of values, and hence there are an infinite number of transformations leaving both A and A′ invariant. The transformations of this system have the property that the resultant of any two of them is another of the same system. Hence they form a continuous group, the parameter of the group being the cross-ratio k, which may be made to vary continuously. We shall designate the group by $G_1(AA')$.

THEOREM 8. The totality of transformations which leave the same two points of a line invariant form a continuous group; the cross-ratio of the resultant of any two transformations of this group is equal to the product of the cross-ratio of the components.

Properties of the group $G_1(AA')$.—The fundamental property of the group $G_1(AA')$ is that the resultant of the two transformations of the group is another of the same group. Other properties of the group will now be developed.

The inverse of T, any transformation in $G_1(AA')$, is also to be found in $G_1(AA')$. For if T transforms x into x_1, then

$$T : \frac{x_1 - A'}{x_1 - A} = k\frac{x - A'}{x - A}.$$

If T^{-1} be the inverse transformation which transforms x_1 back to x, then

$$T^{-1} : \frac{x - A'}{x - A} = \frac{1}{k}\frac{x_1 - A'}{x_1 - A}.$$

Hence the cross-ratios of a pair of inverse transformations have reciprocal values.

The resultant of a pair of inverse transformations is the identical transformation. The product of the cross-ratios of a pair of inverse transformations is unity; hence the cross-ratio of the identical transformation is unity, and the group $G_1(AA')$ always contains the identical transformation.

The group $G_1(AA')$ contains one transformation which is identical with it own inverse. In this case we have the condition $k = \frac{1}{k}$, or $k^2 = 1$; whence $k = \pm 1$. The value $k = 1$ gives the identical trans-

formation of the group. That this is its own inverse is self-evident. The value $k = -1$ gives the involutoric transformation of the group. This transformation has the effect of interchanging every pair of corresponding points on the line, since its second power is the identical transformation; thus this transformation gives rise to an involution, whence its name.

The group $G_1(AA')$ contains two very noteworthy transformations whose cross-ratios are o and ∞, respectively. The first transforms all points of the line except A into A′; the second transforms all points of the line except A′ into A. These are pseudo-transformations and form an inverse pair.

The cross-ratio of the identical transformation is unity, and this transformation leaves every point of the line invariant. The transformation of the group whose cross-ratio is $1 + d$, when d is an infinitesimal number, moves every point on the line an infinitesimal distance, and is hence called an infinitesimal transformation. d has an infinite number of different values, viz., $|\delta| \exp. i\theta$. If an infinitesimal transformation be repeated n times, the cross-ratio of the resultant is $(1 + d)^n$. By a proper choice of d, *i. e.*, of θ and n, this cross-ratio may be made any number we please; hence every transformation in $G_1(AA')$ may be generated from an infinitesimal transformation of the group. The chief properties of the group $G_1(AA')$ may be summed up as follows:

THEOREM 9. The transformations of the group $G_1(AA')$ can be arranged in inverse pairs; it contains the identical transformation, one involutoric, two pseudo and an infinite number of infinitesimal transformations; every transformation of the group may be generated from an infinitesimal transformation of the group.

One-parameter group $G_1(A)$.—Let T and T_1 be two transformations of type II having the same invariant point A. They may be written:

$$\frac{1}{x_1 - A} = \frac{1}{x - A} + t \text{ and } \frac{1}{x_2 - A} = \frac{1}{x_1 - A} + t_1.$$

T transforms x to x_1, and T_1 transforms x_1 to x_2. Their resultant, T_2, is obtained by eliminating x_1 from these two equations by addition, giving us:

$$\frac{1}{x_2 - A} = \frac{1}{x - A} + t + t_1.$$

Thus, $t_2 = t + t$. The resultant, T_2, is of type II, has the same invariant point A, and its constant, t_2, is equal to the sum of the constants of T and T_1.

The constant, t, may have an infinite number of values, and hence there are an infinite number of transformations of type II having the

same invariant point. These possess the same fundamental group property and form a continuous group $G_1(A)$ of type II.

THEOREM 10. The totality of transformations of type II, which leave the same point invariant, form a continuous group; the constant of the resultant of any two transformations of the group is equal to the sum of the constants of the components.

Properties of the group $G_1(A)$.—The properties of the group $G_1(A)$ are not identical with those of the group $G_1(AA')$ but very similar. If T be the transformation

$$\frac{1}{x_1 - A} = \frac{1}{x - A} + t,$$

its inverse, T^{-1}, which transforms x back to x_1, is

$$\frac{1}{x - A} = \frac{1}{x_1 - A} - t.$$

Hence two transformations in $G_1(A)$ whose constants are numerically equal with opposite signs form an inverse pair. All transformations in the group may be arranged in inverse pairs.

The resultant of a pair of inverse transformations is the identical transformation whose constant is $t_2 = t - t = o$. The group $G_1(A)$ therefore contains the identical transformation.

The only transformation in the group which is its own inverse is the identical transformation, *i. e.*, the group contains no involutoric transformation. It contains one pseudo-transformation for which $t = \infty$. This transforms every point on the line to the invariant point.

A transformation of the group whose constant $t = \delta \exp. i\theta$ is infinitesimally near to zero is an infinitesimal transformation. If an infinitesimal transformation is repeated n times, the resultant has the constant nt. By a proper choice of n and θ this may be made any number we please; hence every transformation in the group $G_1(A)$ can be generated from an infinitesimal transformation of the group.

THEOREM 11. The transformation of the group $G_1(A)$ can be arranged in inverse pairs; it contains the identical transformation, one pseudo but no involuntoric transformation; it contains an infinite number of infinitesimal transformations, and every transformation of the group can be generated from an infinitesimal transformation of the group.

Number of one-parameter groups.—We have thus found two types of one-parameter groups of transformations of the points on a line, viz., $G_1(AA')$ and $G_1(A)$. Evidently there are as many groups of the first type as there are pairs of points on a line, viz., ∞^2. Also, there is a group of type II for every point on a line; therefore, ∞^1 in number. It is also evident that every transformation of the points on the line belongs to one and only one of these one-parameter groups (except the identical transformation which is common to all).

§ 4. Two- and Three-parameter Groups.

The group $G_2(A)$.—Let us take two transformations of type I, T and T_1, having one, but only one, invariant point, A′, in common, and find their resultant. The point A′ may be taken for the origin, and the two equations then reduce to the form

$$\frac{x_1}{x_1 - A} = k\frac{x}{x - A} \text{ and } \frac{x_2}{x_2 - A_1} = k_1\frac{x_1}{x_1 - A_1}. \tag{27}$$

Eliminating x_1 from these equations, we have the resultant

$$x_2 = \frac{kk_1 AA_1 x}{(kk_1 A - kA + kA_1 - A_1)x + AA_1}. \tag{28}$$

The invariant points of this transformation are found by making $x_2 = x$ and solving the resulting quadratic. The invariant points are thus found to be $A = o$, and

$$A_2 = \frac{(kk_1 - 1)AA_1}{kk_1 A - kA + kA_1 - A_1}. \tag{29}$$

Putting $kk_1 A - kA + kA_1 - A_1 = \frac{(kk_1 - 1)AA_1}{A_2}$ in equation (28), we have:

$$\frac{x_2}{x_2 - A_2} = kk_1\frac{x}{x - A_2}. \tag{30}$$

From this we see that the resultant of two transformations of type I having one invariant point in common has for one of its invariant points the common invariant point of the components; also, we learn that the cross-ratio of the resultant in this case is equal to the product of the cross-ratios of the components, viz., $k_2 = kk_1$. Since (30) is of the same form as (27), we infer that all transformations leaving a single point invariant form a continuous group of two parameters, $G_2(A)$. The two parameters are the cross-ratio k and the abscissa of the other invariant point A_2.

Theorem 12. All transformations which leave a single point invariant form a two-parameter group; the cross-ratio of the resultant of any two transformations of the group is equal to the product of the cross-ratios of the components.

Structure and properties of $G_2(A)$.—The structure of the group $G_2(A)$ is evident from the above discussion. It contains ∞^1 one-parameter subgroups of type I and one of type II. The point A_2 may be taken in turn with every other point on the line to form the invariant points of a group $G_1(AA')$ and once with itself to form the invariant point of $G_1(A)$. From the continuity of the point system on a line and from the known continuity of each subgroup, we infer the continuity of the group $G_2(A)$. The transformations of the group $G_2(A)$ are not commutative. Since $k_2 = kk_1$, it is evident that the cross-ratio of the resultant is independent of the order of the components; but the position of the second invariant point of T_2 is not independent of the order of T and T_1. For if k and k_1 be inter-

changed in equation (29), the value of A_2 is not changed; but if A and A_1 are interchanged in (29), the value of A_2 is changed, thus showing that T and T_1 are not commutative in $G_2(A)$.

When T and T_1 have both invariant points in common and $k_1 = \frac{1}{k}$, their resultant is the identical transformation; but when T and T_1 have only one invariant point in common and $k_1 = \frac{1}{k}$, the resultant is of type II. For, putting $k_1 = \frac{1}{k}$ in (29), we find $A_2 = o$; thus the two invariant points of T_2 coincide, and it is of type II. The value of the constant t of T_2 is found as follows:

$$t = \lim_{k_2 = 1} \frac{k_2 - 1}{A_2} = \lim \frac{kk_1 A - kA + kA_1 - A_1}{AA_1} =$$
$$\left(\frac{1}{A_1} - \frac{1}{A}\right)(1 - k). \qquad (31)$$

THEOREM 13. The group $G_2(A)$ contains ∞^1 subgroups $G_1(AA')$ and one subgroup $G_1(A)$. The transformations in $G_2(A)$ are not commutative. The resultant of two transformations of type I in $G_2(A)$, for which $k_1 = \frac{1}{k}$ and A'_1 not equal to A' is of type II.

The three-parameter group G_3.—It was shown (Theorem 1) that the resultant of T and T_1, any two projective transformations of the points on a line, is again a projective transformation. From this we infer that all projective transformations of the points on a line form a group. This is called the general projective group G_3. It is a group of three parameters; for the equation of T contains three independent parameters, viz., a : b : c : d. If these coefficients, a, b, c, d, be made to vary continuously, all the resulting transformations belong to the group G_3; and conversely all transformations belonging to the above group are obtained by continuously varying the coefficients in T. Such a group is evidently continuous. If the equation of T be put into the normal form,

$$\frac{x_1 - A'}{x_1 - A} = k \frac{x - A'}{x - A},$$

the three natural parameters, A, A', k, may be made to vary continuously, thus generating the group G_3. The group G_3 contains ∞^1 two-parameter groups $G_2(A)$, one for each point on the line. It contains, as we have already shown, ∞^2 groups $G_1(AA')$ and ∞^1 groups $G_1(A)$.

§ 5. TRANSFORMATIONS OF PENCILS OF LINES AND PLANES.

The theory sketched in the foregoing pages applies equally well to the one-dimensional transformations of the lines of a flat pencil or the planes of an axial pencil. There are two varieties of such transformations, viz., those with two invariant elements and those with only one invariant element.

In the first case let O be the vertex of a flat pencil, A and A' the two

invariant lines of the pencil, and x and x_1 any pair of corresponding lines in the transformation. Then we have the cross-ratio $O(AA'xx_1) = k$, and the theory requires no further development. The second case, with one invariant element, may be deduced as the limiting form of the first case in the following manner:

Let $O(AA'xx_1) = k$; whence $O(AxA'x_1) = 1 - k$. Writing out the last cross-ratio in full, we have:

$$\frac{\sin(A'OA)}{\sin(A'Ox)} : \frac{\sin(x_1OA)}{\sin(x_1Ox)} = 1 - k.$$

Whence

$$\frac{\sin(x_1Ox)}{\sin(A'Ox) \cdot \sin(x_1OA)} = \frac{1-k}{\sin(A'OA)}.$$

But $(xOx_1) = (AOx) - (AOx_1)$; therefore,

$$\lim_{A'=A} \frac{\sin(AOx)\cos(AOx_1) - \cos(AOx)\sin(AOx_1)}{\sin(A'Ox) \cdot \sin(x_1OA)} = \lim_{A'=A} \frac{1-k}{\sin(A'OA)} = t.$$

Hence

$$\cot(x_1OA) - \cot(xOA) = t,$$

$$\text{or} \quad \cot\theta_1 = \cot\theta + t. \qquad (32)$$

THEOREM 14. In a transformation of a pencil of lines (or planes) of type II, the difference of the cotangents of the angles made with the invariant line (or plane) by a pair of corresponding lines (or planes) is constant for all pairs of corresponding lines (or planes).

PART II.

§ 6. THE GENERAL PROJECTIVE GROUP H_6 AND ITS SUBGROUPS.

The complex double parameter.—Hitherto we have considered the natural parameters, A, A′, k of T to be complex numbers, but have regarded them as single indivisible parameters. However, a complex variable is of the nature of a double parameter and may assume ∞^2 different values. From this point of view, the groups which we have designated by $G_1(AA')$ and $G_1(A)$, whose parameters are respectively k and t, are two-parameter groups. In like manner, $G_2(A)$ and G_3 have, respectively, four and six parameters. The first point of view is that of Lie and his followers, while the second point of view is more that of Klein and other exponents of Riemann's function theory.

According to Lie's theory of continuous groups, there are but four varieties of continuous groups of transformations in one dimension, viz., G_3, $G_2(A)$, $G_1(AA')$, $G_1(A)$. But as soon as we regard a complex parameter as a double parameter, we shall find several other varieties of continuous groups within the general projective group of points on the line. These will be determined in the next section.

Hereafter we shall use H instead of G to designate a group whose parameters are regarded as double parameters, and shall write H_6, $H_4(A)$, $H_2(AA')$, $H_2(A)$ instead of G_3, $G_2(A)$, $G_1(AA)$, $G_1(A)$.

These two sets of symbols represent exactly the same things, but from different points of view.

Loxodromic, hyperbolic, elliptic and parabolic transformations.—We have already shown that the transformation $T: x_1 = \frac{ax+b}{cx+d}$ is reducible to one or the other of the normal forms:

$$\frac{x_1 - A'}{x_1 - A} = k\frac{x - A'}{x - A} \text{ or } \frac{1}{x_1 - A} = \frac{1}{x - A} + t.$$

Transformations of the first type are subdivided into three kinds, according to the values of k. When k is real,* the transformation is called hyperbolic; when k is of the form $\exp i\theta$, *i. e.*, when $|k| = 1$, it is called elliptic; for all the other values of k it is called loxodromic. All transformations of the second type are called parabolic. These distinctions will be of much use to us.

Known subgroups of H_6.—From § 4 we know that H_6 contains ∞^2 subgroups $H_4(A)$, one for each value of A. H_6 also contains ∞^4 subgroups $H_2(AA')$, one for each pair of values of A and A'. It also contains ∞^2 groups $H_2(A)$, one for each value of A. The ∞^4 parabolic transformations in H_6 do not form a group. This follows from the fact that the resultant of two parabolic transformations is not generally a parabolic transformation.

One-parameter subgroups of $H_2(A)$.—In the group $H_2(A)$ the law of combination of parameters is $t_2 = t + t_1$; the parameter t may be written in the form, $r \exp i\theta$. Thus we have

$$r_2 \exp i\theta_2 = r \exp i\theta + r_1 \exp i\theta_1.$$

If $\theta_1 = \theta$, then $\theta_2 = \theta$. Keeping θ constant and letting r vary, we get $r_2 = r + r_1$.

We have here the conditions for a one-parameter subgroup of $H_2(A)$. All transformations in $H_2(A)$ for which the t's have the same amplitude form a subgroup of $H_2(A)$. It is clear that there is one such subgroup for each value of θ.

If $t = r\exp i\theta$ be represented in the usual manner by points in the complex plane, we see that there is a transformation in $H_2(A)$ corresponding to each point in the complex plane. The transformations in $H_2(A)$, corresponding to the points of the complex plane on a line through the origin and making an angle θ with the axis of reals, constitute a subgroup $H_1(A)\theta$.

Properties of $H_1(A)\theta$.—The group $H_1(A)\theta$ contains the identical transformation corresponding to $r = o$. Two transformations corresponding to values of r numerically equal, but with opposite signs, are inverse to one another. $H_1(A)\theta$ also contains the pseudo-

*Klein (Modulfunctionen, Band I, S. 164), and Poincare (Acta Mathematica, Tome I, p. 5), call the transformation hyperbolic only when k is real and positive. The change here made is ustified by the simplification it effects in the statement of the results.

transformation of $H_2(A)$, which is given by $r = \infty$. This group contains two subdivisions corresponding to positive and negative values of r. These are separated by the identical and the pseudo-transformations.

The group $H_1(A)\theta$ contains two infinitesimal transformations corresponding to $+\delta r$ and $-\delta r$. Each infinitesimal transformation generates its corresponding subdivision of the group. It is evident that each transformation of the group $H_2(A)$ can be generated from one and only one infinitesimal transformation.

One may, if one chooses, call the aggregate of the transformations in $H_2(A)$, which are represented by points on a half ray from the origin to the infinity point, a group. Every such group contains the identical transformation, the pseudo-transformation, one infinitesimal transformation, and every transformation in the group is generated from the infinitesimal transformation of the group. But the inverse of every transformation in such a group $H_1(A)\theta$ would be found in group $H_1(A)(\theta+\pi)$. It is better, however, to consider the group as made up of two subdivisions than to regard each subdivision as a separate group. The group H_6 contains ∞^2 subgroups $H_2(A)$, one for each value of the complex number A. Each of these two-parameter groups $H_2(A)$ contains ∞^1 one-parameter subgroups, one for each real value θ between 0 and π.

THEOREM 15. There are two varieties of subgroups of parabolic transformations in H_6, viz., $H_2(A)$ and $H_1(A)\theta$.

One-parameter subgroups of $H_2(AA')$.—In the group $H_2(AA')$ the law of combination of parameters is $k_2 = kk_1$. The parameter k may be written in the form $\exp(c+i)\theta$; then we have:

$$\exp(c_2+i)\theta_2 = \exp\left\{(c+i)\theta + (c_1+i)\theta_1\right\}.$$

If $c_1 = c$, then $c_2 = c_1 = c$, and $\theta_2 = \theta + \theta_1$. Keeping c fixed and making θ vary, we get $\exp(c+i)\theta_2 = \exp(c+i)(\theta+\theta_1)$. We have here the conditions for a one-parameter subgroup of $H_2(AA')$; all transformations in $H_2(AA')$, for which the k's have the same c in the formula $k = \exp(c+i)\theta$, form a subgroup $H_1(AA')c$. There is one such subgroup for each real value of c.

There is a transformation in $H_2(AA')$ corresponding to each value of k, which is a complex number. Let $k = \exp(c+i)\theta$ be represented by points in a complex plane. We wish to see how the transformations in the one-parameter group $H_1(AA')c$ are distributed in the plane. We have the equation $k = \exp(c+i)\theta$, where c is a constant and θ a variable. Let $k = x + iy$. Then we have:

$$x + iy = \exp c\theta(\cos\theta + i\sin\theta); \qquad (33)$$

$$x = \exp c\theta\cos\theta;$$

hence,

$$y = \exp c\theta\sin\theta;$$

therefore,

$$x^2 + y^2 = \exp 2c\theta;$$

putting $x^2 + y^2 = r^2$, we have $r = \exp c\theta$. But this is the polar equation of a logarithmic spiral about the origin cutting the axis of real numbers at an angle whose cotangent is c. Hence the locus of points in the complex plane which correspond to the transformations in the one-parameter group $H(AA')c$, is a logarithmic spiral about the zero point cutting the axis of reals at an angle ϕ, such that $\cot \phi = c$.

Different values of c give different spirals, each of which corresponds to a one-parameter subgroup of $H_2(AA')$. The real number c may assume in turn all values from $-\infty$ to $+\infty$, so that these spirals lie infinitely close to one another. These spirals all pass through the unit point. For $c = o$, the corresponding spiral becomes the unit circle; for $c = \infty$, the spiral reduces to a straight line, the axis of real numbers.

The family of spirals for which c is positive fills the entire plane, and no two of them intersect except in the unit point. The same is true of the family of spirals for which c is negative. But every spiral of one family intersects an infinite number of times every spiral of the other family; thus these spirals cover twice over the entire plane. Every point in the plane not on the unit circle lies on two of these spirals, from which we infer that every loxodromic transformation in the group $H_2(AA')$ belongs to two distinct loxodromic one-parameter subgroups.

Every hyperbolic transformation in $H_2(AA')$, except the involutoric transformation for which $k = -1$, belongs to three one-parameter subgroups; for two spirals and the axis of real numbers pass through every point for which k is real. The elliptic transformations in $H_2(AA')$ belong only to the elliptic subgroup. The involutoric transformation is common to the elliptic and the hyperbolic subgroups. The identical transformation is common to all subgroups, and the two pseudo-transformations, for which $k = o$ and $k = \infty$, are common to all subgroups except the elliptic. Two loxodromic subgroups for which the c's have the same sign have no transformations in common other than the identical and the pseudo-transformations, while two loxodromic subgroups for which the c's have opposite signs have in common an infinite number of discrete transformations.

THEOREM 16. Every elliptic transformation in $H_2(AA')$ belongs to one and only one subgroup; every loxodromic transformation in $H_2(AA')$ belongs to two distinct subgroups; every hyperbolic transformation in $H_2(AA')$, except the involutoric transformation, belongs to three distinct subgroups.

Generation of $H_2(AA')$ from infinitesimal transformations.—The same geometric representation enables us to discuss intuitively the generation of finite transformations in $H_2(AA')$ by the repetition

of an infinitesimal transformation. Every spiral passes through the unit point, and, corresponding to the two points on a spiral adjacent to the unit point, we have two infinitesimal transformations belonging to a one-parameter group. These are given by $k = \exp(c + i)\delta\theta$ and $k = \exp -(c + i)\delta\theta$. The indentical transformation divides the group $H_1(AA')c$ into two portions, each of which contains an infinitesimal transformation. Every finite transformation in each portion of $H_1(AA)c$ can be generated by the repetition of the corresponding infinitesimal transformation.

Since every loxodromic transformation in $H_2(AA')$ belongs to two distinct subgroups, it follows that every such loxodromic transformation can be generated from either of two distinct infinitesimal transformations. In the elliptic group, for which the spiral reduces to the unit circle, every finite transformation can be generated from either elliptic infinitesimal transformation. Every hyperbolic transformation for which k is positive can be generated from three infinitesimal transformations, while every hyperbolic transformation in $H_1(AA')$ for which k is negative, except the involutoric transformation, can be generated from two distinct loxodromic infinitesimal transformations, but not from either hyperbolic infinitesimal transformation.

THEOREM 17. Every hyperbolic transformation in $H_2(AA')$, for which k is positive, can be generated from three distinct infinitesimal transformations belonging to $H_2(AA')$; every other finite transformation in $H_2(AA')$ can be generated from two distinct infinitesimal transformations of the group.

Three-parameter subgroups of $H_4(A)$.—The four-parameter group $H_4(A)$ breaks up, as we already know, into ∞^2 two parameter subgroups $H_2(AA')$. We shall now show that the transformations in $H_4(A)$ may be distributed into ∞^1 three-parameter subgroups. The law of combination of parameters k within the group $H_4(A)$ is expressed (Theorem 12) by $kk_1 = k_2$. Written in another form, this is

$$\exp(c_2 + i)\theta_2 = \exp\left\{ (c + i)\theta + (c_1 + i)\theta \right\}.$$

If $c_1 = c$, we have $\exp(c_2 + i)\theta_2 = \exp(c + i)(\theta + \theta_1)$; whence $c_2 = c$ and $\theta_2 = \theta + \theta_1$. Hence we see that if we choose from each of the two-parameter groups, $H_2(AA')$, in $H_4(A)$, the one-parameter group characterized by a certain constant value of c, the aggregate of the transformations contained in these one-parameter groups forms a three-parameter group $H_3(A)c$. It is clear that there is one such group in $H_4(A)$ for every real value of c.

In particular, all elliptic transformations in $H_4(A)$ form a group $eH_3(A)$; likewise, all hyperbolic transformations in $H_4(A)$ form a group $hH_3(A)$.

THEOREM 18. All transformations having a common invariant point A, and for which the value of c in the formula $k = \exp(c + i)\theta$ is the same, form a three-parameter subgroup $H_3(A)c$ of $H_4(A)$; $eH_3(A)$ and $hH_3(A)$ are among the subgroups $H_4(A)$.

§ 7. INTERPRETATION OF H_6 IN COMPLEX PLANE.

The linear fractional transformation,

$$T : z_1 = \frac{az + b}{cz + d}, \tag{34}$$

in which the coefficients and variables are complex quantities, is capable of interpretation as a point transformation of the complex plane. We proceed to the development of the transformation from this point of view.

T is a circular transformation.—We shall first show that the transformation T transforms circles into circles. The equation of a real circle in Cartesian coordinates is

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = o. \tag{35}$$

Let $x_1 + iy_1 = z$, $x_1 - iy_1 = \bar{z}$, $g + if = \alpha$, $g - if = \beta$. Making these substitutions in (34), we get as the equation of the circle:

$$z_1\bar{z}_1 + \beta z_1 + \alpha\bar{z}_1 + \gamma = o. \tag{36}$$

α and β are conjugate complex quantities; so, also, are z_1 and $\bar{z}_1$, but γ is real.

If we change the sign of i throughout the equation of T, we have:

$$\bar{z}_1 = \frac{\bar{a}\,\bar{z} + \bar{b}}{\bar{c}\,\bar{z} + \bar{d}}, \tag{37}$$

where the quantities under the dash are the conjugates of those without the dash. Substituting (34) and (37) in (36) and clearing, we have:

$$Kz\bar{z} + Lz + M\bar{z} + N = o, \tag{38}$$

where

$$K = a\bar{a} + \beta\, a\bar{c} + \alpha\, \bar{a}c + \gamma\, c\bar{c},$$
$$L = a\bar{b} + \beta\, a\bar{d} + \alpha\, c\bar{b} + \gamma\, c\bar{d},$$
$$M = \bar{a}b + \beta\, \bar{c}b + \alpha\, \bar{a}d + \gamma\, \bar{c}d,$$
$$N = \bar{b}b + \beta\, b\bar{d} + \alpha\, \bar{b}d + \gamma\, \bar{d}d.$$

On examination, K and N are found to be real and L and M conjugate imaginary. These are the necessary and sufficient conditions that equation (38) represents a circle. Thus circles are transformed into circles by T, which is, therefore, called a circular transformation.

Straight lines are transformed into circles.—It may happen for certain values of a, b, c, d that K in equation (38) vanishes. In this case (38) represents a straight line. If we substitute (34) and (37) in $Ax + By + C = o$, we find that this line is transformed into a circle.

Hence, circles are sometimes transformed into a straight line, and straight lines usually into circles.

In the geometry of the complex plane, a straight line is regarded as a special case of a circle, and every circle passing through the one point at infinity of the complex plane is a straight line. Regarding a straight line as a particular case of a circle, our transformation always transforms circles into circles.

THEOREM 19. A linear fractional transformation in complex variables is a circular transformation of the complex plane.

T is a conformal transformation.—Another fundamental property of our transformation T is that angles are transformed into equal angles. To show this, consider a small triangle formed by the points z, z', z'', and compare it with the triangle formed by their corresponding points z_1, z'_1, x''_1. In $\frac{z'_1 - z_1}{z''_1 - z_1}$ substitute their values from (34); we get:

$$\frac{z'_1 - z_1}{z''_1 - z_1} = \frac{cz'' + d}{cz' + d} \cdot \frac{z' - z}{z'' - z}. \tag{39}$$

The ratio $\frac{cz'' + d}{cz' + d}$ tends to unity as the triangles become very small. The quantities $z'_1 - z_1$, $z' - z$, etc., are the corresponding sides of the two triangles; hence, when the corresponding triangles are very small, the corresponding sides are proportional and the triangles are similar. Angles are therefore transformed into equal angles. A transformation which leaves angular magnitudes unaltered is called a *conformal* transformation.

THEOREM 20. A linear fractional transformation of the complex plane is a conformal transformation.

T is a direct circular transformation.—The two triangles, z, z', z'' and z_1, z'_1, z''_1, are not only in the limit similar triangles, but they are also congruent triangles. The homologous sides are arranged in the same order around the two triangles, and if they were of the same size the two triangles could be made to coincide.

Suppose that the three sides of the triangle z, z', z'' are colored red, green, and blue, and that an observer at z, facing the opposite side, has on his right the red, on his left the blue, and before him the green. The transformation T changes the triangle z, z', z'' into the new triangle z_1, z'_1, z''_1, and the observer at his new position at z_1 still finds the red on his right and the blue on his left. Such a transformation is a direct circular transformation.

There are other transformations which are both circular and conformal, but which interchange the parts of the triangle, so that in his new position the observer finds the blue on his right and the red on his left. These are called indirect circular transformations.

THEOREM 21. A linear fractional transformation of the complex plane is a direct circular transformation.

Path curves of the group $H_1(AA')c$.—The effect of the successive transformations of the group $H_1(AA)c$ upon a point P of the complex plane is to move it to new positions, P_1, P_2, P_3, etc., which lie on the path curve of the point P. By a transformation $T(AA')c$ every point of the plane is moved along its path curve. The character and properties of these path curves are now to be determined.

Starting with normal form,

$$\frac{z_1 - A'}{z_1 - A} = \exp(c + i)\theta \frac{z - A'}{z - A},$$

for a given value of θ, z is transformed into z_1; for another value of θ, z is transformed into z_2, etc. The point z may thus be regarded as a fixed point, and $\frac{z - A'}{z - A} = C$, a constant. Let z′ denote the moving point z_1, z_2, etc. We thus get:

$$z' - A' = C \exp(c + i)\theta\, z' - A.$$

Let $z'_1 - A = r' \exp i\phi'$, $C = \rho \exp i\psi$, and $z' - A = r \exp i\phi$; we then have

$$r' \exp i\phi' = \rho \exp i\psi \,.\, \exp(c + i)\theta \,.\, r \exp i\phi.$$

Whence,

$$r' = \rho \exp i\theta\, r \text{ and } \phi' = \psi + \theta + \phi. \qquad (40)$$

These two equations enable us to determine the path curves of the group $H_1(AA')c$.

In the first place, let $c = o$; we then have $r' = \rho r$, where ρ is a constant. The locus of a point z, which moves so that the ratio of its distances from two fixed points A and A′ is a constant, is a circle having A and A′ for a pair of inverse points. Hence, the path curves of the elliptic group $eH_1(AA')$ form an elliptic system of coaxial circles having A and A′ for vanishing points.

In the second place, let $\exp(c + i)\theta \equiv \rho \exp i\theta$ be real; whence $\theta = 0$. We now have $\phi' - \phi = \psi$, where ψ is a constant angle. The locus of a point z, which moves so that the angle AZA′ is constant, is a circle passing through A and A′. Hence, the path curves of the hyperbolic group $hH_1(AA')$ form a hyperbolic system of coaxial circles intersecting in A and A′.

In the third place, let us consider the most general case where c in $\exp(c + i)\theta$ is any real constant. In the equation $r' = \rho \exp c\theta\, r$, the quantity $\rho \exp c\theta$ is not constant, but increases as θ increases from o to 2π, from 2π to 4π, etc. Thus the locus of z is not a circle returning into itself, but a spiral about both A and A′. It is known as the double spiral of Holzmüller. The equation, $\phi' - \phi = \psi + \theta$, shows that the spiral cuts the hyperbolic system of circles through A and A′ at a constant angle ψ. Hence the path curves of the loxodromic group $H_1(AA')c$ form a system of equiangular double spirals about A and

A′. There is a different system of spirals for each value of c. For $c = o$ and $c = \infty$ this system of spirals reduces, respectively, to elliptic and hyperbolic systems of circles.

THEOREM 22. The path curves of a one-parameter loxodromic group $H_1(AA')c$ form a system of double spirals about the invariant points A and A′; the path curves of $eH_1(AA')$ and $hH_1(AA')$ are, respectively, elliptic and hyperbolic systems of circles about and through A and A′.

Path curves of $H_1(A)\theta$.—The path curves of the one-parameter parabolic group $H_1(A)\theta$ may be determined in the following manner: The normal form of a parabolic transformation is

$$\frac{1}{z_1 - A} = \frac{1}{z - A} + t. \tag{41}$$

Suppose that z is a fixed point and z_1 a movable point whose locus we wish to find. Equation (41) may be written:

$$z_1 - z = -t\,(z - A)\,(z_1 - A).$$

Putting

$$(z_1 - z) = R' \exp i\phi, \quad -t = \rho \exp i(\theta + \pi), \quad (z - A) = \exp i\psi,$$
$$(z_1 - A) = R \exp i\phi,$$

this becomes

$$R' \exp i\phi' = \rho \exp i(\theta + \pi) \,.\, r \exp i\psi \,.\, R \exp i\phi.$$

Whence we have $\quad R' = \rho r R$ and $\phi' = \theta + \pi + \phi + \psi. \qquad (42)$

Since ψ, θ and π are constants, we have $\phi' - \phi -$ const. Thus we see that the locus of z_1 is a circle passing through A and z, two fixed points.

If now z be given different positions in the plane, we have a system of circles all passing through A. The angle ψ varies with the position of z, but θ is a constant; consequently, all circles of the system are tangent at A, and θ is the angle which the common tangent line at A makes with the axis of real numbers. Hence the path curves of the group $H_1(A)\theta$ consist of a parabolic system of circles tangent to each other at A and to the line through A which makes with the axis of reals an angle θ.

THEOREM 23. The path curves of the one-parameter group $H_1(A)\theta$ consist of a parabolic system of circles through A and having in common the lineal element $A\theta$.

§ 8. OTHER SUBGROUPS OF H_6.

The group $hH_2(AC)$.—The developments of § 6 enable us to find some other subgroups of great importance in H_6. Let us consider the hyperbolic group $hH_3(A)$; it contains, as we know, ∞^2 subgroups $hH_1(AA')$. The resultant of any two transformations, $hT(AA')$ and $hT_1(AA'_1)$, in $hH_3(A)$ is a hyperbolic transformation $hT_2(AA'_2)$. We

know that $k_2 = kk_1$. We wish to find the position of the point A'_2. The path curves of the two groups $hH_1(AA')$ and $hH_1(AA'_1)$ have one circle in common, viz., C, the circle through the points A, A′, A'_1. This circle is therefore invariant under the resultant transformation $T_2(AA'_2)$. Hence, A'_2 is somewhere on the circle C. The system of transformations which leave A and C invariant have the group property and form a two-parameter group $hH_2(AC)$. This group contains ∞^1 one-parameter hyperbolic subgroups, one for each point on C, and one parabolic subgroup, $H_1(A)\theta$, which has C among its path curves.

The group $H_3(C)$.—There is a two-parameter group $hH_2(AC)$ corresponding to each point on C. These ∞^3 transformations all leave C invariant, but they do not form a group, as we shall show. There are also ∞^3 elliptic transformations which leave C invariant. Let A be any point within C, and A′ its inverse point with respect to C. The one-parameter group $eH_1(AA')$ has C among its system of path curves. In like manner all one-parameter groups of elliptic transformations, whose invariant points are a pair of invariant points with respect to C, leave C invariant. There are ∞^2 such pairs of points, and hence there are ∞^3 elliptic transformations leaving C invariant.

The aggregate of all transformations leaving C invariant forms a three-parameter group $H_3(C)$. This group contains ∞^3 hyperbolic, ∞^3 elliptic and ∞^2 parabolic transformations. It breaks up into subgroups as follows:

$$H_3(C) \equiv \infty^1 H_2(AC) + \infty^2 eH_1(AA') \equiv \infty^2 hH_1(AA') + \infty^2 eH_1(AA') + \infty^1 H'_1(A),$$

Evidently the group H_6 contains ∞^3 subgroups of the kind $H_3(C)$, one for each circle in the complex plane.

THEOREM 24. All transformations of the complex plane, which leave a circle C invariant, form a three-parameter group $H_3(C)$. This group is composed of all hyperbolic transformations whose invariant points are on the circle, of all elliptic transformations whose invariant points are a pair of inverse points with respect to the circle, and of all parabolic transformations whose invariant point is on the circle and whose invariant line is tangent to the circle at the invariant point.

The group $H_3(iC)$.—There is another type of the three-parameter group consisting entirely of elliptic transformations, which is closely related to the group $H_3(C)$. This group of transformations of the complex plane leaves invariant an imaginary circle (iC). The relation of this group $H_3(iC)$ to the group $H_3(C)$ with real invariant circle is as follows:

The elliptic system of coaxial circles which form the path curves of a one-parameter elliptic group contains imaginary as well as real circles. Let us select from the path curves of the group $eH_1(AA')$ an imaginary circle (iC), with its center at a real point O and radius equal to iR. The center O is on the line AA′, between A and A′, and the points A and A′ are a pair of inverse points with respect to iC; whence we have $OA \,.\, OA' = -R^2$. In the case of a real circle C, we have $OA \,.\, OA' = R^2$. All transformations in the group $eH_1(AA')$ leave invariant the circle iC and also ∞^1 other circles.

There are ∞^2 pairs of points in the plane which are inverse points with respect to iC. These all satisfy the relation $OA \,.\, OA' = -R^2$. Each of these pairs of points are the invariant points of a one-parameter group of elliptic transformations, and iC is one of the path curves of each of these groups. Thus we see that there are ∞^3 transformations which leave iC invariant; these form a three-parameter group $H_3(iC)$, which contains only elliptic transformations. Evidently the group H_6 contains ∞^3 subgroups $H_3(iC)$, one for each imaginary circle in the complex plane.

THEOREM 25. There are ∞^3 transformations of the complex plane which leave invariant any given imaginary circle iC; these form a group $H_3(iC)$. This group is composed entirely of elliptic transformations whose invariant points are a pair of inverse points with respect to (iC).

No other subgroups of H_6.—A circular transformation transforms points into points and circles into circles. We have considered all possible groups which leave one or two points invariant; a transformation leaving invariant more than two points is identical. We have also considered all possible groups of transformations leaving a circle invariant. If there be a continuous group characterized by the invariance of some curve other than a circle, such a curve must be the path curve of a one-parameter group. The only other path curve besides the circle is the double spiral of Holzmüller. This has two singular points, and is invariant only under those transformations whose invariant points are these two singular points; hence, there is only one one-parameter group leaving invariant such a double spiral. These considerations indicate that there are no other subgroups of H_6.

§ 9. THE REAL SUBGROUP OF H_6.

RH_3 a special case of $H_3(C)$.—In the geometry of the complex plane a straight line is regarded as a special case of a circle, and since every circle in the plane is the invariant circle of a three-parameter group $H_3(C)$, it follows that every line in the plane is also invariant under a group $H_3(L)$, isomorphic with $H_3(C)$. One of these groups $RH_3(L)$ leaves invariant the axis of real numbers in the complex

plane. It is called the real subgroup of H_6. The equation of $RH_3(L)$ is

$$x_1 = \frac{ax+b}{cx+d}, \tag{43}$$

in which the variable and coefficients are all real quantities. The structure and properties of $RH_3(L)$ are identical with those of $H_3(C)$.

THEOREM 26. The three-parameter group of real projective transformations of points on a real line is a special subgroup of the six-parameter group of circular transformations of the complex plane.

Properties of $RH_3(L)$.—The properties of the real projective group in one dimension, which we shall now designate by RG_3, may be determined in two ways. They may be deduced from the properties of the group $H_3(C)$, or they may be derived directly from equation (43). The analytical developments of §§ 1, 2, 3, 4 apply equally well to equation (1) whether the coefficients a, b, c, d are real or complex. From the known properties of $H_3(C)$, given in § 8, we may deduce the following for RG_3:

The real group of projective transformations RG_3 contains three kinds of transformations, viz., hyperbolic, elliptic, and parabolic. A hyperbolic transformation has two real invariant points; an elliptic, two conjugate imaginary invariant points; a parabolic, one real invariant point. RG_3 breaks up into ∞^1 subgroups $RG_2(A)$, one for each real point on the line. It further breaks up into one-parameter subgroups, as follows: $\infty^2 hRG_1(AA')$, $\infty^2 eRG_1(AA')$, and $\infty^1 pRG_1(A)$. The natural parameters of $hRG_1(AA')$ are A, A′ and k, which are all real. Those of $eRG_1(AA')$ are A, A′ and k where A and A′ are conjugate imaginary and k is of the form $\exp i\theta$. The parameters of $pRG_1(A)$ are A and t, both real.

Generation of real groups from infinitesimal transformations.—The generation of the real groups of one-dimensional transformations requires special notice. We consider first the group $hRG_1(AA')$. The natural parameter of this group is k, which is always a real number. The identical transformation of the group is given by $k = 1$. The two transformations of the group given by $k = \pm d$, where d is a real, positive, infinitesimal quantity, are infinitesimal transformations. The transformations of $hRG_1(AA')$, for which k is between 1 and ∞, may be generated from the infinitesimal transformation for which $k = 1 + d$; for $(1+d)^n$ may be made any positive number greater than unity by proper choice of n. In like manner we see that the transformations of the group for which k is between 1 and 0 may be generated by a repetition of the infinitesimal transformation for which $k = 1 - d$. The transformations in $hRG_1(AA')$ for which k is negative cannot be generated from either

infinitesimal transformation of the group. They may be generated from complex infinitesimal transformations but not from real.

Hence, we distinguish three subdivisions of $hRG_1(AA')$: those transformations of the group for which k is positive and less than 1 constitute subdivision I; those for which k is between 1 and ∞ constitute subdivision II; those for which k is negative constitute subdivision III.

THEOREM 27. The group $hRG_1(AA')$ is composed of three subdivisions characterized by $0<k<1$, $1<k<\infty$, $k<0$; subdivisions I and II contain each a generating infinitesimal transformation; the transformations in subdivision III cannot be generated from either infinitesimal transformation of the group.

Properties of the group $eGH_1(AA')$.—In an elliptic subgroup of RG_3 the natural parameter k has the form of $\exp i\theta$. The value of θ may be restricted to the interval $-\pi<\theta<\pi$. The identical and involutoric transformations of the group for which $\theta=\pm\pi$, *i. e.*, $k=-1$, divide the group into two subdivisions. Each subdivision contains an infinitesimal transformation given by $\theta=\pm\delta$ where δ is an infinitesimal real number; every transformation in the group may be generated from either infinitesimal transformation of the group.

THEOREM 28. The group $eRG_1(AA')$ contains two infinitesimal transformations given by $\theta=\pm\delta$, each of which may generate the entire group.

Properties of the group $pRG_1(A)$.—The parabolic group $pRG_1(A)$ has the real parameter t, and the law of combination of parameters is $t_2=t+t_1$. Since the identical transformation is given by $t=o$, it follows that the group has two infinitesimal transformations given by $+\delta$ and $-\delta$. Each infinitestimal transformation generates its corresponding portion of the group, which, therefore, is composed of two subdivisions. For one subdivision t is positive; for the other, negative.

THEOREM 29. The group $pRG_1(A)$ is composed of two subdivisions, each of which contains its generating infinitesimal transformation.

TABLE OF CONTINUOUS GROUPS IN ONE DIMENSION.

In the following table the fourteen varieties of continuous groups of one-dimensional transformations are classified according to their types, and the structure of each is shown.

LOXODROMIC GROUPS.

First class.

Symbol.	Structure.	Invariant figure.
$H_2(AA')$	$=H_2(AA')+hH_1(AA')+eH_1(AA')$.	Two points.
$H_4(A)$	$=\infty^2H_2(A\underline{A}'+pH_2(A)+hH_3(A)+eH_3(A)$.	One point.
H_6	$=\infty^4H_2(\underline{A}\,\underline{A}')+\infty^2pH_2(\underline{A})+\infty^2hH_3(\underline{A})+\infty^2eH_3(\underline{A})$.	No invariant figure.

Second class.

$H_1(AA')_c$.	Two points and a system of double spirals.
$H_3(A)_c = \infty^2 H_1(A\underline{A'})_c + pH_2(A)$.	One point.
HYPERBOLIC GROUPS.	
$hH_1(AA')$.	Two points and a hyperbolic system of circles.
$H_2(AC) = \infty^1 hH_1(A\underline{A'}) + pH_1(A)_0$.	Circle and point on it.
$hH_3(A) = \infty^2 hH_1(A\underline{A'}) + pH_2(A)$.	One point.
$H_3(C) = \infty^2 hH_1(\underline{A}\,\underline{A'}) + \infty^2 eH_1(\underline{A}\,\underline{A'}) + \infty^1 pH_1(A)_0$.	Circle.
ELLIPTIC GROUPS.	
$eH_1(AA')$	Two points and an elliptic system of circles.
$eH_3(A) = \infty^2 eH_1(A\underline{A'}) + pH_2(A)$.	One point.
$H_3(iC) = \infty^2 eH_1(\underline{A}\,\underline{A'})$.	Imaginary circle.
PARABOLIC GROUPS.	
$pH_1(A)_0$.	One point and a parabolic system of circles.
$pH_2(A) = \infty^1 pH_2(A)_0$.	One point.